

Application of higher order holonomy corrections to perturbation theory of cosmology

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Applying the higher order holonomy corrections to the perturbation theory of cosmology, the lattice power law of Loop Quantum Cosmology, $\tilde{\mu} \propto p^\beta$, is analysed and the range of β is decided to be $[-1, 0]$ which is different from the conventional range $-0.1319 > \beta \geq -5/2$ [1]. At the same time, we find that there is a anomaly free condition in this theory, and we obtain this condition in the vector and tensor mode. We also find that the nonzero mass of gravitational wave essentially results from the quantum nature of Riemannian geometry of loop quantum gravity.

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I. INTRODUCTION

The spacetime metric of Big Bang cosmology is homogeneous and isotropic Friedmann-Robertson-Walker (FRW) metric. However, this model is just an approximation of “zero order” universe [2]. If we only focus on the FRW metric, we will ignore many of interesting things in the universe such as galaxy clusters, galaxies, stars, etc. So it is necessary to introduce the inhomogeneous and anisotropy perturbation to describe these things [3]. On the other hand, the effects of quantum gravity should be significant in the very early universe. Therefore, it is interesting to study possible quantum gravity effects in cosmological perturbation theory.

At present, the problem of finding the quantum theory of the gravitational field is still open. One of the most active of the current approaches is loop quantum gravity. Loop quantum gravity (LQG) [4–6] is a mathematically well-defined, non-perturbative and background independent quantization of general relativity. Its cosmological version, the loop quantum cosmology (LQC) [7] have achieved many successes. A major success of LQC is the resolution of the Big Bang singularity [8, 9, 12]; this result depends crucially on the discreteness of the spacetime geometry. With such a result, the big-bang singularity will be avoided through a big-bounce mechanism in the high energy region. In addition, LQC can also setup suitable initial conditions for successful inflation [13, 14] as well as possibly leaving an imprint in the cosmic microwave background [14].

In LQG, spacetime is quantized. The geometric operators, such as the area operator and the volume operator, have discrete eigenvalues. So there is the smallest area gap Δ [15, 16]. In LQC, the coordinate size of a loop is $\tilde{\mu}^2$. $\tilde{\mu}$ is the function of $p = a^2$ (where a is the scale factor of the universe.), i.e. $\tilde{\mu} = \tilde{\mu}(p)$. In the early literature [9, 10], the work always base on the simplest choice of $\tilde{\mu}(p) = \mu_0 = \text{const}$. However, this form can lead to some unusual features. As pointed out in [12] that the choice

of $\tilde{\mu}(p) = \text{const}$ can lead to the Big Bounce occurs at classical matter density like water, so he suggest to select the function as $\tilde{\mu}(p) \propto p^{-1/2}$. From this time forth, in most of current works [11], $\tilde{\mu}(p) \propto p^{-1/2}$ has been applied. And it was shown that the choice $\tilde{\mu}(p) \propto p^{-1/2}$ is physically and mathematically consistent [12]. Up to now, however, there is still no theory to decide the function of $\tilde{\mu}(p)$. As research continues, there may be some other form of $\tilde{\mu}(p)$ can give the better physics. Therefore, to find out the form of this function has theoretical significance.

An ansatz for the form of this function can be taken as $\tilde{\mu}(p) \propto p^\beta$. In [1], the range of β has been decided to be $-0.1319 > \beta \geq -5/2$. However, it is just the conclusion of the first order holonomy corrections. If we want a more accurate determination of the range of β , we must consider the higher order corrections.

Even in the case of homogeneous and isotropic models, the quantum equation of state is very difficult to analyze. Fortunately, there is a powerful tool, i.e. effective theory, which allows us to include loop quantum effects by correction terms in equations of the classical type [17]. There are two types of quantum corrections that are expected from the Hamiltonian of LQG. One correction arises for inverse powers of the densitized triad, which when quantized becomes an operator with zero in the discrete part of its spectrum thus lacking a direct inverse. The other comes from the fact that a loop quantization is based on holonomies, i.e. exponentials of the connection rather than direct connection components [18].

In LQC, there is no well-defined quantum operator corresponding to $c = \gamma k$. So we should find a well-defined operator to replace it. The conventional way is replacing the c by $\sin \tilde{\mu} c / \tilde{\mu}$.

The application of inverse triad corrections and conventional holonomy corrections on the scalar mode of perturbation can be viewed in [19], the vector mode in [20] and the tensor mode in [1].

In this paper, we focus on the higher holonomy corrections rather than the conventional correction. We apply these higher corrections to the vector and the tensor mode and see whether the mass of gravitational wave is the nature of discrete geometry. We will also analyses the range of β with high order holonomy corrections.

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This paper is organized as follows. At first, the perturbed variables are introduced in Sec. II. And then in Sec. III, we apply the high order holonomy corrections to obtain the effective Hamiltonian constraint. Detailed analysis of the range of β will be given in Sec. IV. Section V is our discussion.

II. BACKGROUND AND PERTURBED CONSTRAINT

In Ashtekar's formalism of general relativity [21, 22], the spatial metric as a canonical field is replaced by the densitized triad E_i^a , defined as

$$E_i^a := \left| \det(e_b^j) \right| e_i^a, \quad (1)$$

and the spin connection Γ_a^i which is

$$\Gamma_a^i = -\epsilon^{ijk} e_j^b \left(\partial_{[a} e_{b]}^k + \frac{1}{2} \epsilon_k^c e_a^l \partial_{[c} e_{b]}^l \right). \quad (2)$$

The canonical variables are densitized triad E_i^a and Ashtekar connection $A_a^i = \Gamma_a^i + \gamma K_a^i$, where K_a^i is extrinsic curvature and γ is Barbero-Immirzi parameter.

The canonical variables reduced to spatially homogeneous and isotropic cosmology are

$$E_i^a = p \delta_i^a, \quad K_a^i = k \delta_a^i, \quad \Gamma_a^i = 0. \quad (3)$$

They are background variables, and the perturbation will be added based on these variables.

In the perturbation theory, we denote the background variables by a bar:

$$\bar{E}_i^a = \bar{p} \delta_i^a, \quad \bar{\Gamma}_a^i = 0, \quad \bar{K}_a^i = \bar{k} \delta_a^i, \quad \bar{N} = \sqrt{\bar{p}}; \quad \bar{N}^a = 0, \quad (4)$$

where $\bar{p} = a^2$, and the spatial metric is $\bar{q}_{ab} = a^2 \delta_{ab}$. We use conformal time in this paper, so we set $\bar{N} = a$.

The canonical variables are the perturbation densitized triad E_i^a and Ashtekar connection A_a^i , which are

$$E_i^a = \bar{p} \delta_i^a + \delta E_i^a, \quad A_a^i = \bar{\Gamma}_a^i + \gamma K_a^i = \gamma \bar{k} \delta_a^i + (\delta \Gamma_a^i + \gamma \delta K_a^i), \quad (5)$$

where δE_i^a and δK_a^i are small perturbation around homogeneous variables.

As described in [1, 19, 20], the symplectic structure splits into two parts: one for the background variables and the other for perturbations, i.e.,

$$\{\bar{k}, \bar{p}\} = \frac{8\pi G}{3V_0}, \quad (6)$$

and

$$\{\delta K_a^i(x), \delta E_j^b(y)\} = 8\pi G \delta^3(x, y) \delta_a^b \delta_j^i. \quad (7)$$

Here, G is the gravitational constant and V_0 is a fiducial volume.

In vector mode, the gravity part of perturbed Hamiltonian constraint (up to quadratic terms) is [20]

$$H_G[N] = \frac{1}{16\pi G} \int_{\Sigma} d^3x \bar{N} \left[\bar{k}^2 \left(-6\sqrt{\bar{p}} - \frac{\delta E_j^c \delta E_k^d \delta_c^k \delta_d^j}{2\bar{p}^{3/2}} \right) + \sqrt{\bar{p}} (\delta K_c^j \delta K_d^k \delta_c^c \delta_j^d) - \frac{2\bar{k}}{\sqrt{\bar{p}}} (\delta E_j^c \delta K_c^j) \right]. \quad (8)$$

On the other hand, when we introduce the inhomogeneous perturbation, the diffeomorphism constraint does not vanish any more. So the gravitational part of diffeomorphism constraint is changed into [20]

$$D_G[N^a] = \frac{1}{8\pi G} \int_{\Sigma} d^3x \delta N^c \left[-\bar{p} (\partial_k \delta K_c^k) - \bar{k} \delta_c^k (\partial_d \delta E_k^d) \right]. \quad (9)$$

Using Eqs.(6) and (7), we can testify the following relation easily

$$\{H_G, D_G\} = 0. \quad (10)$$

Similarly, in tensor mode, the gravity part of the perturbed Hamiltonian constraint (up to quadratic terms) is [1]

$$H_G[N] = \frac{1}{16\pi G} \int_{\Sigma} d^3x \bar{N} \left[\bar{k}^2 \left(-6\sqrt{\bar{p}} - \frac{\delta E_j^c \delta E_k^d \delta_c^k \delta_d^j}{2\bar{p}^{3/2}} \right) + \sqrt{\bar{p}} (\delta K_c^j \delta K_d^k \delta_c^c \delta_j^d) - \frac{2\bar{k}}{\sqrt{\bar{p}}} (\delta E_j^c \delta K_c^j) + \frac{1}{\bar{p}^{3/2}} (\delta_{cd} \delta^{jk} \delta^{ef} \partial_e E_j^c \partial_f E_k^d) \right], \quad (11)$$

where $\delta E_i^a = -\frac{1}{2} \bar{p} h_i^a$, here $h_a^i := \delta^{ib} h_{ab}$, and h_{ab} is the symmetric metric perturbation field. It is transverse and traceless, i.e. it satisfies $\partial^a h_{ab} = 0$ and $\delta^{ab} h_{ab} = 0$ [3].

III. VECTOR AND TENSOR MODE WITH HIGHER ORDER HOLONOMY CORRECTIONS

A. Higher order holonomy corrections

Instead of the conventional way of introducing the holonomy corrections, in this article, we focus on the higher order holonomy corrections [23].

At first, let's consider the Taylor series

$$\sin^{-1} x = \sum_{l=0}^{\infty} \frac{(2l)!}{2^{2l} (l!)^2 (2l+1)} x^{2l+1} \quad (12)$$

for $-1 \leq x \leq 1$ and setting $x = \sin(\tilde{\mu} \gamma \bar{k})$, we have

$$\gamma \bar{k} = \frac{1}{\tilde{\mu}} \sum_{l=0}^{\infty} \frac{(2l)!}{2^{2l} (l!)^2 (2l+1)} [\sin(\tilde{\mu} \gamma \bar{k})]^{2l+1}, \quad (13)$$

This inspires us to define a n th order holonomized connection variable as

$$c_h^{(n)} := \frac{1}{\tilde{\mu}} \sum_{l=0}^n \frac{(2l)!}{2^{2l} (l!)^2 (2l+1)} [\sin(\tilde{\mu}\gamma\bar{k})]^{2l+1}, \quad (14)$$

which can be made arbitrarily close to $\gamma\bar{k}$ as $n \rightarrow \infty$. We can see that $c_h^{(n)}$ is a function of the holonomy $\sin(\tilde{\mu}\gamma\bar{k})$ and the discreteness variable $\tilde{\mu}$. Therefore, we can replace $\gamma\bar{k}$ by $c_h^{(n)}$ to implement the underlying structure of LQC. When $n = 0$, $c_h^{(0)} = \sin(\tilde{\mu}\gamma\bar{k})/\tilde{\mu}$ is the same with the conventional holonomy corrections.

There is an ambiguity in this replacement. If we set $x = \sin(m\tilde{\mu}\gamma\bar{k})$ in Eq.(12), where m is an arbitrary constant, Eq.(13) changes to

$$m\gamma\bar{k} = \frac{1}{\tilde{\mu}} \sum_{l=0}^{\infty} \frac{(2l)!}{2^{2l} (l!)^2 (2l+1)} [\sin(m\tilde{\mu}\gamma\bar{k})]^{2l+1}, \quad (15)$$

and we have

$$\gamma\bar{k} = \frac{1}{m\tilde{\mu}} \sum_{l=0}^{\infty} \frac{(2l)!}{2^{2l} (l!)^2 (2l+1)} [\sin(m\tilde{\mu}\gamma\bar{k})]^{2l+1}. \quad (16)$$

So we can define a more general n th order holonomized connection variable $c_{mh}^{(n)}$:

$$c_{mh}^{(n)} := \frac{1}{m\tilde{\mu}} \sum_{l=0}^n \frac{(2l)!}{2^{2l} (l!)^2 (2l+1)} [\sin(m\tilde{\mu}\gamma\bar{k})]^{2l+1}, \quad (17)$$

where m is an ambiguity parameter.

The Poisson brackets between the canonical variables and the $c_h^{(n)}$ are

$$\left\{ \bar{p}, \frac{c_h^{(n)}}{\gamma} \right\} = -\frac{8\pi G}{3V_0} \cos(\tilde{\mu}\gamma\bar{k}) \mathfrak{G}_n(\tilde{\mu}\gamma\bar{k}), \quad (18)$$

and

$$\left\{ \bar{k}, \frac{c_h^{(n)}}{\gamma} \right\} = \frac{8\pi G}{3\tilde{\mu}V_0} \frac{\partial \tilde{\mu}}{\partial \bar{p}} \left[\cos(\tilde{\mu}\gamma\bar{k}) \mathfrak{G}_n(\tilde{\mu}\gamma\bar{k}) \bar{k} - \frac{c_h^{(n)}}{\gamma} \right], \quad (19)$$

where

$$\mathfrak{G}_n(\tilde{\mu}\gamma\bar{k}) = \sum_{l=0}^n \frac{(2l)!}{2^{2l} (l!)^2} [\sin(\tilde{\mu}\gamma\bar{k})]^{2l}. \quad (20)$$

From [23] we can see that the role of higher order holonomy corrections is like a filter, which excludes the impact of human factors on the theory and leaves a pure quantum effect.

B. Vector mode

In classical perturbation theory of cosmology, the gauge invariant variables of the vector mode will decay

quickly. Therefore, there is a little role of the vector mode perturbation for a universe [3]. However, once we introduce the quantum correction, we must consider whether the perturbation theory is anomaly free [20]. The requirement of anomaly free can reduce some ambiguities of LQC. Inserting the higher holonomy corrections in Eq.(8), we can obtain the effective gravity part of the perturbed Hamiltonian constraint

$$\begin{aligned} H_G^Q[N] = & \frac{1}{16\pi G} \int_{\Sigma} d^3x \bar{N} \left\{ \left(\frac{c_h^{(n)}}{\gamma} \right)^2 \right. \\ & \times \left[-6\sqrt{\bar{p}} - \frac{1}{2\bar{p}^{3/2}} \left(\delta E_j^c \delta E_k^d \delta_c^k \delta_d^j \right) \right] \\ & \left. + \sqrt{\bar{p}} \left(\delta K_c^j \delta K_d^k \delta_c^c \delta_j^d \right) - \frac{2}{\sqrt{\bar{p}}} \frac{c_{mh}^{(n)}}{\gamma} \delta E_j^c \delta K_c^j \right\}. \end{aligned} \quad (21)$$

General speaking, we should replace all the $\gamma\bar{k}$ by $c_{mh}^{(n)}$. But in order to get a homogeneous limit which agreement with what has been used in isotropic models, we set the parameter m in the first term to equal one [1]. The parameter m in the last term should lead to an anomaly-free constraint algebra, so we do not fix it at first. In the following discussion, we will determine the right value of m in the last term by requiring an anomaly-free constraint algebra in the presence of quantum corrections.

In homogeneous and isotropic model, there is no diffeomorphism constraint. So the algebra of constraints is closed. When we consider the inhomogeneous perturbation, the diffeomorphism constraint will turn up. From Eq.(10) we can see that, in classical theory, the algebra of constraints is still closed. So when we write down the constraints with the quantum corrections, we need to ensure that the constraints are still closed. In other words, the anomaly terms, which cannot be expressed by the linear combination of the Hamiltonian constraint and the diffeomorphism constraint, should be vanished. On the other hand, the diffeomorphism constraint does not receive quantum corrections in the full theory [24], so Eq.(9) does not change.

The Poisson bracket between two constraints is

$$\begin{aligned} & \{H_G^Q, D_G\} \\ = & \frac{\bar{N}}{\sqrt{\bar{p}}} \left[\bar{k} + \frac{c_{mh}^{(n)}}{\gamma} - 2 \frac{c_h^{(n)}}{\gamma} \cos(\tilde{\mu}\gamma\bar{k}) \mathfrak{G}_n(\tilde{\mu}\gamma\bar{k}) \right] D_G \\ & + \frac{1}{8\pi G} \int_{\Sigma} d^3x \bar{p} (\partial_c \delta N^j) \mathcal{A}_j^{(n)c}, \end{aligned} \quad (22)$$

where the anomaly part is

$$\begin{aligned} \mathcal{A}_j^{(n)c} = & \frac{\bar{N}}{\sqrt{\bar{p}}} \left\{ \bar{p} \frac{\partial}{\partial \bar{p}} \left(\frac{c_h^{(n)}}{\gamma} \right)^2 + \left(\frac{c_h^{(n)}}{\gamma} \right)^2 - \bar{k}^2 \right. \\ & \left. + 2 \left[\frac{c_h^{(n)}}{\gamma} \cos(\tilde{\mu} \gamma \bar{k}) \mathfrak{G}_n(\tilde{\mu} \gamma \bar{k}) - \frac{c_{mh}^{(n)}}{\gamma} \right] \bar{k} \right\} \left(\frac{\delta E_j^c}{\bar{p}} \right). \end{aligned} \quad (23)$$

To get this, we need the Poisson bracket

$$\{\delta K_c^j(x), \partial_d \delta E_k^d(y)\} = 8\pi G \delta_c^j \delta_c^d \partial_d \delta(x, y), \quad (24)$$

$$\{\delta E_c^j(x), \partial_d \delta K_k^d(y)\} = -8\pi G \delta_c^j \delta_c^d \partial_d \delta(x, y). \quad (25)$$

To cancel the anomaly part, it must be requested that $\mathcal{A}_j^{(n)c} = 0$, i.e.

$$\begin{aligned} \frac{c_{mh}^{(n)}}{\gamma \bar{k}} = & (\beta + 1) \frac{c_h^{(n)}}{\gamma \bar{k}} \cos(\tilde{\mu} \gamma \bar{k}) \mathfrak{G}_n(\tilde{\mu} \gamma \bar{k}) \\ & + \frac{1 - 2\beta}{2} \left(\frac{c_h^{(n)}}{\gamma \bar{k}} \right)^2 - \frac{1}{2}. \end{aligned} \quad (26)$$

From [23] we know that the big bounce occur when $\tilde{\mu} c = \frac{\pi}{2}$, so the maximum of $c_{mh}^{(n)} \tilde{\mu}$ is $\frac{\pi}{2}$. According Eq.(26), we have

$$(\beta + 1) c_h^{(n)} \tilde{\mu} \cos(\tilde{\mu} \gamma \bar{k}) \mathfrak{G}_n(\tilde{\mu} \gamma \bar{k}) + \left[\frac{1 - 2\beta}{2} \left(\frac{c_h^{(n)}}{\gamma \bar{k}} \right)^2 - \frac{1}{2} \right] \tilde{\mu} \gamma \bar{k} \leq \frac{\pi}{2}. \quad (27)$$

Eq.(27) can be seen as a limit to the evolution of $c_{mh}^{(n)}$, and we can restrict the range of β through this limit.

C. Tensor mode

In classical perturbation theory of cosmology, there is only one equation in tensor mode, i.e. the gravitational waves equation. From this equation, we know that the gravitational waves are massless. However, when quantum corrections are taken into account, a mass term will be appeared in this equation [1]. It is only the conclusion calculated in the first order correction. We extend this method to the higher holonomy corrections, and take limit of $n \rightarrow \infty$. In this way, we can find that the mass of gravitational waves is the intimately results from the quantum nature of Riemannian geometry of LQG.

Inserting the higher holonomy corrections to Eq.(11), the effective gravity part of perturbed Hamiltonian constraint can be expressed as

$$\begin{aligned} H_G^Q[N] = & \frac{1}{16\pi G} \int_{\Sigma} d^3x \bar{N} \left\{ \left(\frac{c_h^{(n)}}{\gamma} \right)^2 [-6\sqrt{\bar{p}} \right. \\ & - \frac{1}{2\bar{p}^{3/2}} (\delta E_j^c \delta E_k^d \delta_c^k \delta_d^j)] + \sqrt{\bar{p}} (\delta K_c^j \delta K_d^k \delta_c^c \delta_k^d) \\ & \left. - \frac{2}{\sqrt{\bar{p}}} \frac{c_{mh}^{(n)}}{\gamma} \delta E_j^c \delta K_c^j + \frac{\delta_{cd} \delta^{jk} \delta^{ef} \partial_e E_j^c \partial_f E_k^d}{\bar{p}^{3/2}} \right\}. \end{aligned} \quad (28)$$

From this Hamiltonian, one can obtain the time deriva-

tive of the background variables

$$\bar{p} = 2\bar{p} \frac{c_h^{(n)}}{\gamma} \cos(\tilde{\mu} \gamma \bar{k}) \mathfrak{G}_n(\tilde{\mu} \gamma \bar{k}) \quad (29)$$

$$\begin{aligned} \bar{k} = & -\frac{1}{2} \left(\frac{c_h^{(n)}}{\gamma} \right)^2 - 2 \frac{c_h^{(n)}}{\gamma} \frac{\bar{p}}{\tilde{\mu}} \frac{\partial \tilde{\mu}}{\partial \bar{p}} \\ & \times \left[\cos(\tilde{\mu} \gamma \bar{k}) \mathfrak{G}_n(\tilde{\mu} \gamma \bar{k}) \bar{k} - \frac{c_h^{(n)}}{\gamma} \right], \end{aligned} \quad (30)$$

and the time derivative of the perturbed variable δE_i^a

$$\delta \dot{E}_i^a = \left\{ \delta E_i^a, H_G^Q \right\} = -\bar{p} \delta_k^a \delta_i^d \delta K_d^k - \frac{1}{2} \bar{p} \frac{c_{mh}^{(n)}}{\gamma} h_i^a. \quad (31)$$

On the other hand, one can also obtain $\delta \dot{E}_i^a$ from $\delta E_i^a = -\frac{1}{2} \bar{p} h_i^a$, i.e.

$$\begin{aligned} \delta \dot{E}_i^a = & -\frac{1}{2} (\dot{\bar{p}} h_i^a + \bar{p} \dot{h}_i^a) \\ = & -\frac{1}{2} \left(2\bar{p} \frac{c_h^{(n)}}{\gamma} \cos(\tilde{\mu} \gamma \bar{k}) \mathfrak{G}_n(\tilde{\mu} \gamma \bar{k}) h_i^a + \bar{p} \dot{h}_i^a \right). \end{aligned} \quad (32)$$

From Eqs.(31) and (32), we have

$$\delta K_a^i = \frac{1}{2} \left[\dot{h}_a^i + \left(2 \frac{c_h^{(n)}}{\gamma} \cos(\tilde{\mu} \gamma \bar{k}) \mathfrak{G}_n(\tilde{\mu} \gamma \bar{k}) - \frac{c_{mh}^{(n)}}{\gamma} \right) h_a^i \right]. \quad (33)$$

So, the $\delta\dot{K}_a^i$ will be

$$\delta\dot{K}_a^i = \frac{1}{2} \left[\ddot{h}_a^i + h_a^i \partial_t \left(2 \frac{c_h^{(n)}}{\gamma} \cos(\tilde{\mu}\gamma\bar{k}) \mathfrak{G}_n(\tilde{\mu}\gamma\bar{k}) - \frac{c_{mh}^{(n)}}{\gamma} \right) + \left(2 \frac{c_h^{(n)}}{\gamma} \cos(\tilde{\mu}\gamma\bar{k}) \mathfrak{G}_n(\tilde{\mu}\gamma\bar{k}) - \frac{c_{mh}^{(n)}}{\gamma} \right) \dot{h}_a^i \right]. \quad (34)$$

Again, we can also obtain $\delta\dot{K}_a^i$ from Hamiltonian equation

tion

$$\begin{aligned} \delta\dot{K}_a^i &= \left\{ \delta K_a^i, H_G^Q \right\} + \left\{ \delta K_a^i, H_{matter} \right\} \\ &= \left\{ \delta K_a^i, H_{matter} \right\} + \frac{1}{4} \left(\frac{c_h^{(n)}}{\gamma} \right)^2 h_a^i - \frac{1}{2} \frac{c_{mh}^{(n)}}{\gamma} \dot{h}_a^i + \frac{1}{2} \nabla^2 h_a^i \\ &\quad - \frac{1}{2} \frac{c_{mh}^{(n)}}{\gamma} \left(2 \frac{c_h^{(n)}}{\gamma} \cos(\tilde{\mu}\gamma\bar{k}) \mathfrak{G}_n(\tilde{\mu}\gamma\bar{k}) - \frac{c_{mh}^{(n)}}{\gamma} \right) \dot{h}_a^i. \end{aligned} \quad (35)$$

From Eqs.(34) and (35), one can obtain the gravitational waves equation

$$\frac{1}{2} \left[\ddot{h}_a^i + 2 \frac{c_h^{(n)}}{\gamma} \cos(\tilde{\mu}\gamma\bar{k}) \mathfrak{G}_n(\tilde{\mu}\gamma\bar{k}) \dot{h}_a^i - \frac{1}{2} \nabla^2 h_a^i + T_Q^{(n)} h_a^i \right] = 8\pi G \Pi_{Qa}^i, \quad (36)$$

where Π_{Qa}^i is the source terms from the matter Hamiltonian and

$$\begin{aligned} T_Q^{(n)} &= -2 \frac{\partial \tilde{\mu}}{\partial \bar{p}} \frac{\bar{p}}{\tilde{\mu}} \left\{ 2 \tilde{\mu}^2 \gamma^2 \left[\left(\frac{c_h^{(n)}}{\gamma} \right)^4 \mathfrak{G}_n(\tilde{\mu}\gamma\bar{k}) - \left(\frac{c_h^{(n)}}{\gamma} \right)^3 \frac{\cos^2(\tilde{\mu}\gamma\bar{k})}{\gamma \tilde{\mu}} \mathfrak{B}_n(\tilde{\mu}\gamma\bar{k}) \right] \right. \\ &\quad \left. - \frac{c_h^{(n)}}{\gamma} \left[\cos(\tilde{\mu}\gamma\bar{k}) \mathfrak{G}_n(\tilde{\mu}\gamma\bar{k}) \frac{c_{mh}^{(n)}}{\gamma} - \cos(m\tilde{\mu}\gamma\bar{k}) \mathfrak{G}_n(m\tilde{\mu}\gamma\bar{k}) \frac{c_h^{(n)}}{\gamma} \right] \right\} \\ &\quad + \frac{1}{2} \left(\frac{c_h^{(n)}}{\gamma} \right)^2 \left\{ 2 \frac{c_h^{(n)}}{\gamma} \gamma \tilde{\mu} [\sin(\tilde{\mu}\gamma\bar{k}) \mathfrak{G}_n(\tilde{\mu}\gamma\bar{k}) - \cos^2(\tilde{\mu}\gamma\bar{k}) \mathfrak{B}_n(\tilde{\mu}\gamma\bar{k})] \right. \\ &\quad \left. - 1 + \cos(m\tilde{\mu}\gamma\bar{k}) \mathfrak{G}_n(m\tilde{\mu}\gamma\bar{k}) \right\} - \left(\frac{c_h^{(n)}}{\gamma} \cos(\tilde{\mu}\gamma\bar{k}) - \frac{c_{mh}^{(n)}}{\gamma} \right)^2, \end{aligned} \quad (37)$$

where

$$\mathfrak{B}_n(\tilde{\mu}\gamma\bar{k}) = \sum_{l=0}^n \frac{2l(2l!)}{2^{2l}(l!)^2} [\sin(\tilde{\mu}\gamma\bar{k})]^{2l-1}. \quad (38)$$

When $n = 0$, $\mathfrak{B}_n(\tilde{\mu}\gamma\bar{k}) = 0$. So it will not appear in conventional way. The definition of the effective mass is

$$m_g^2 := \frac{T_Q}{a^2}. \quad (39)$$

When $\tilde{\mu}\gamma\bar{k} \rightarrow \frac{\pi}{2}$, $n \rightarrow \infty$, the mass term will never vanish. From this, we can confirm that the nonzero mass of gravitational wave results from the quantum nature of Riemannian geometry of LQG.

From the definition of gravitational wave [1], we should require $T_Q^{(n)} \geq 0$, and this is another condition to restrict the range of β .

IV. LATTICE REFINEMENTS

In process of obtaining the range of β in [1], the author expand the “sin” (and “cos”) of Eq.(26)($n = 0$) and just

take the first several terms of it. So in his paper, the first non-zero term of anomaly part is k^4 . By this way, one can obtain the relationship between m and β from Eq.(26)($n = 0$), and obtain the range of β by requiring the $T_Q^{(n)} > 0$ from Eq.(37)($n = 0$).

However, there are some problems in this method. First of all, the relation of $m^2 = 5 + 2\beta$ in [1] is obtain by requiring the term of k^4 is vanished. But it can not ensure the whole anomaly part can be canceled, because there are still $k^6, k^8, etc.$ Secondly, when we consider the higher corrections, there will be some high order terms like $\sin^3(x)$ appear. If we keep more terms of “sin” (and “cos”), we will find that the first non-zero term is not k^4 , maybe, it will be k^5 , it depend on how many terms you kept. It decides the different relation between m and β . So if we want to obtain the range of β more accurate, we should not expand the “sin”(and “cos”) in the equations, in other words, we keep all the terms of it.

From the discussions above we can know that, Eq.(27) and $T_Q^{(n)} \geq 0$ are two restricts to β , so we analyze these two restricts respectively. At first, we note that, the product $\tilde{\mu}\gamma\bar{k}$ always appear together in the expression of

$c_{mh}^{(n)}\tilde{\mu}$, so we set $x = \tilde{\mu}\gamma\bar{k}$, and then $c_{mh}^{(n)}\tilde{\mu}$ is the function of x . When Big Bounce occurs at $x = \frac{\pi}{2}$, and $x \rightarrow 0$ with the expansion of the universe, we can draw the graphs of this function between 0 to $x = \frac{\pi}{2}$ with different value of n .

From Fig.1 we can see that, when $n = 0$ (it correspond the conventional holonomy corrections), the case of $\beta =$

$-\frac{5}{2}$ (which was lower bound in [1]) can fill the condition of $c_{mh}^{(n)}\tilde{\mu} \leq \frac{\pi}{2}$. But it can be more lower than that because $\beta = -2.7$ can also fill the condition. However, we will be concerned about the higher corrections, so let us analyze the case of large n . From Fig.1, we can see that, the larger n lead to the bigger lower bound of β .

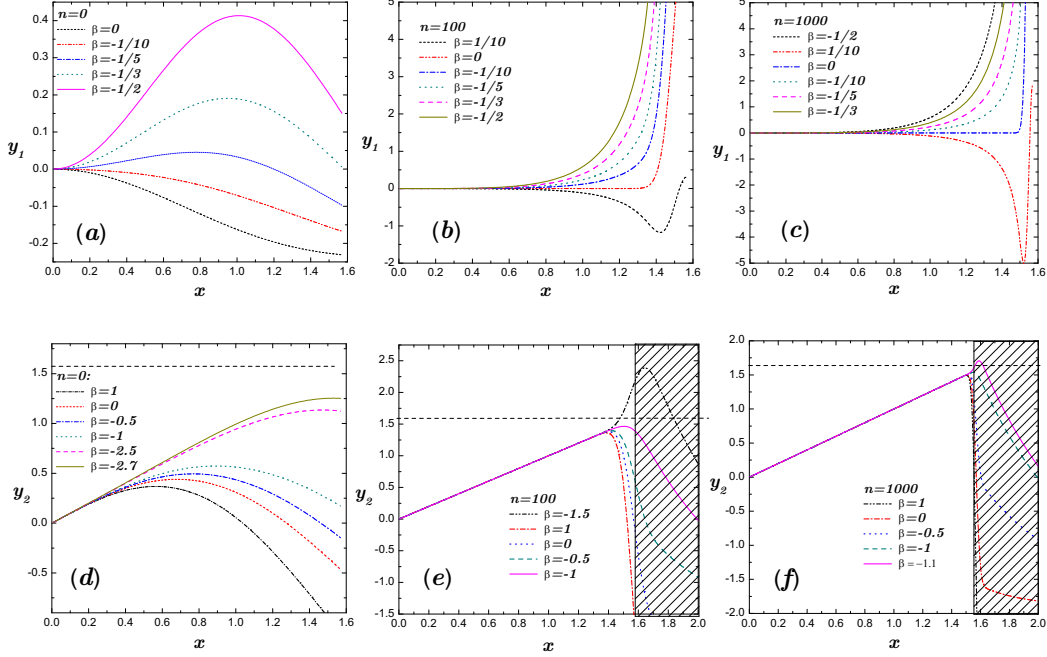


FIG. 1: The evolution of $y_1 = \frac{T_Q^{(n)}}{k^2}$ and $y_2 = c_{mh}^{(n)}\tilde{\mu}$ with $x = \tilde{\mu}\gamma\bar{k}$ are shown in [(a),(b),(c)] and [(d),(e),(f)] respectively. The dash lines in (d),(e)and (f) are $y_2 = \frac{\pi}{2}$.

Because the Eq.(27) should be kept everywhere, it includes the point of the Big Bounce i.e. $x = \frac{\pi}{2}$. Inserting $x = \frac{\pi}{2}$ to (27), it will become

$$\left(\frac{1-2\beta}{2} - \frac{1}{2}\right) \frac{\pi}{2} \leq \frac{\pi}{2}, \quad (40)$$

which leads to

$$\beta \geq -1. \quad (41)$$

On the other hand, the Eq.(26) is the relationship of m and β . We insert the $\frac{c_{mh}^{(n)}}{\gamma k}$ into Eq.(37). On the right hand of Eq.(37), there are still some terms which contain m , so we can use the following relation to replace this

terms

$$\cos(m\tilde{\mu}\gamma\bar{k}) \mathfrak{G}_n(m\tilde{\mu}\gamma\bar{k}) = \tilde{\mu}\gamma\bar{k} \frac{\partial}{\partial(\tilde{\mu}\gamma\bar{k})} \frac{c_{mh}^{(n)}}{\gamma} + \frac{c_{mh}^{(n)}}{\gamma}. \quad (42)$$

From Eqs.(26), (37) and (42) we can see that $\frac{T_Q^{(n)}}{k^2}$ is also the function of $x = \tilde{\mu}\gamma\bar{k}$, and there is two parameters n and β in it.

From the evolution of $\frac{T_Q^{(n)}}{k^2}$ with $n = 0$ (see Fig.1(a)), we can see that, if we require the $\frac{T_Q^{(n)}}{k^2} > 0$, the β should be smaller than $-\frac{1}{3}$. With Eq.(41), the range of β is $-\frac{1}{3} > \beta \geq -1$. This result is smaller than $-0.1319 > \beta \geq -5/2$. It is because we use the “sin”, not the first

orders of expanding term of “sin”.

When $n > 0$, we find that the upper bound of β is larger than $-\frac{1}{3}$, and when $n \rightarrow \infty$, the upper bound will be zero. We display the $n = 100$ and $n = 1000$ in Fig.1 also.

So, the final range of β should be $[-1, 0]$. From this we can see that $\beta = 0$ is not eliminated like in [1] and $\beta = -\frac{1}{2}$ is also in this range.

V. DISCUSSION

In this paper, we apply the higher order holonomy corrections to the perturbation theory of cosmology. When we take the limit of $n \rightarrow \infty$, the form of the LQC will be back to classical theory, but the effect of the quantum geometry will be kept. From the analyses above, we know that the mass of gravitational wave will not be vanished when $n \rightarrow \infty$. It will decrease to zero with the expansion of the universe. So it is the “pure” quantum effect that the gravitational wave have nonzero mass.

Other important effects are related to the discrete space-time geometry. Discrete space means the existence of the area gap, and there is a function $\tilde{\mu}(p)$ related to this area gap. The form of function $\tilde{\mu}(p)$ has an important impact on LQC. But now the framework of theory is not perfect to decide this function, so we only can restrict the form of the function as $\tilde{\mu} \propto p^\beta$, from some other aspects like effective theory and perturbation theory of cosmology.

In the effective LQC framework, we apply two conditions to limit the range of β . One is anomaly free, which means that the constraint algebra of vector mode should be closed, when we consider the the quantum effect. It is

the mathematical requirements of the theory. This can restrict β to be $[-1, +\infty)$.

The other condition is the requirements of positive definite mass of gravitational waves. This is the physical requirement. We can not ensure that the mass of gravitational wave is positive when $\beta > 0$. And from Fig.1, we can see that the behavior of mass between $\beta > 0$ and $\beta \leq 0$ is very different. Therefore it can restrict the range of β to be $(-\infty, 0]$. This requirement seems very natural. However, we do not yet understand the true meaning of the mass of gravitational waves, so this condition is only an assumption. The correctness of this assumption needs to be verified in future studies.

In conclusion, the range of β should be $[-1, 0]$. But this range is only decided by perturbation theory of cosmology. It cannot exclude $\beta = 0$. So the excluding of $\beta = 0$ is based on the prediction of theory rather than theory itself. This may not be its final scope because only two conditions were discussed in this article. Certainly there are many other conditions to limit the range of parameter. If we can restrict β to a unique value, say $-1/2$, from the theory itself rather than predictive power of theory, then the theory will be more self-consistency. So, in the future studies, we can compare different conditions on the parameter values to examine the self-consistency of theory.

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